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Stochastic flows on metric graphs*

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Abstract

We study a simple stochastic differential equation (SDE) driven by one Brownian motion on a general oriented metric graph whose solutions are stochastic flows of kernels. Under some conditions, we describe the laws of all solutions. This work is a natural continuation of [17, 8, 10] where some particular metric graphs were considered.

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1 Introduction

A metric graph is seen as a metric space with branching points. In recent years, diffusion processes on metric graphs are more and more studied [7, 12, 13, 14, 15]. They arise in many physical situations such as electrical networks, nerve impulsion propagation [5, 18]. They also occur in limiting theorems for processes evolving in narrow tubes [4]. Diffusion processes on graphs are defined in terms of their infinitesimal operators in [6]. Such processes can be described as mixtures of motions "along an edge" and "around a vertex". Typical examples of such processes are Walsh's Brownian motions introduced in [20] and defined on a finite number of half lines which are glued together at a unique end point. These processes have acquired a particular interest since it was proved by Tsirelson [19] that they cannot be strong solutions to any SDE driven by a standard Brownian motion, although they satisfy a martingale representation property with respect to some Brownian motion [1]. In view of this, it is natural to investigate SDEs on graphs driven by one Brownian motion to be as simple as possible. This study has been initiated by Freidlin and Sheu in [6] where any Walsh's Brownian motion X has been shown to satisfy the equation

$$df(X_t) = f'(X_t)dW_t + \frac{1}{2}f''(X_t)dt$$

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where W is the Brownian motion given by the martingale part of $|X|$, f runs over an appropriate domain of functions with an appropriate definition of its derivatives. Our subject in this paper is to investigate the following extension on a general oriented metric graph :

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}f'(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du$$

where K is a stochastic flow of kernels as defined in [16], W is a real white noise, f runs over an appropriate domain, f' and f'' are defined according to an arbitrary choice of coordinates on each edge. When G is a star graph, this equation has been studied in [8] and when G consists of only two edges and two vertices the same equation has been considered in [10]. In this paper, we extend these two studies (as well as [17] where the associated graph is simply the real line) and classify the solutions on any oriented metric graph.

The content of this paper is as follows.

In Section 2, we introduce notations for any metric graph G and then define a SDE (E) driven by a white noise W, with solutions of this SDE being stochastic flows of kernels on G (Definition 2.3). Thereafter, our main result is stated. Along an edge the motion of any solution only depends on W and the orientations of the edges. The set of vertices of G will be denoted by V. Around a vertex $v \in V$, the motion depends on a flow $\hat{K}^v$ on a star graph (associated to v) as constructed in [8].

In Section 3, starting from $(\hat{K}^v)_{v\in V}$ respective solutions to a SDE on a star graph associated to a vertex v, we construct a stochastic flow of kernels K solution of (E) under the following additional (but natural) assumption : the family $\left(\vee_{v\in V}\mathcal{F}^{\hat{K}^v}_{s,t};\ s \le t\right)$ is independent on disjoint time intervals (here $\mathcal{F}^{\hat{K}^v}_{s,t}$ is the sigma-field generated by the increments of $\hat{K}^v$ between s and t).

In Section 4, starting from K, we recover the flows $(\hat{K}^v)_{v\in V}$. Actually, in Sections 3 and 4, we prove more general results : the SDEs may be driven by different white noises on different edges of G (see [9] for other applications of these results).

The main results about flows on star graphs obtained in [8] are reviewed in Section 5. Thus, as soon as the flows $(\hat{K}^v)_{v\in V}$ can be defined jointly, we have a general construction of a solution K of (E). In Section 6, we consider two vertices v_1 and v_2 and under some condition only depending on the "geometry" of the star graphs associated to v_1 and v_2, we show that independence on disjoint time intervals of $\left(\mathcal{F}^{\hat{K}^{v_1}}_{s,t} \vee \mathcal{F}^{\hat{K}^{v_2}}_{s,t}, s \le t\right)$ is equivalent to : $\hat{K}^{v_1}$ and $\hat{K}^{v_2}$ are independent given W.

Section 7 is an appendix devoted to the skew Brownian flow constructed by Burdzy and Kaspi in [3]. We will explain how this flow simplifies our construction on graphs such that any vertex has at most two adjacent edges.

Section 8 is an appendix complement to Section 5. Therein, we review the construction of the flows $\hat{K}^v$ constructed in [8] with notations in accordance with the content of our paper.

2 Definitions and main results

2.1 Oriented metric graphs

Let G be a metric graph as defined in [13, Section 2.1] in the sense that there exists a finite or countable set V, the set of vertices, and a partition $\{E_i;\ i \in I\}$ of $G\backslash V$ with I a finite or countable set (i.e. $G\backslash V = \cup_{i\in I}E_i$ and for $i \neq j$, $E_i \cap E_j = \emptyset$) such that for all $i \in I$, E_i is isometric to an interval $(0, L_i)$, with $L_i \le +\infty$. We call E_i an edge, L_i its

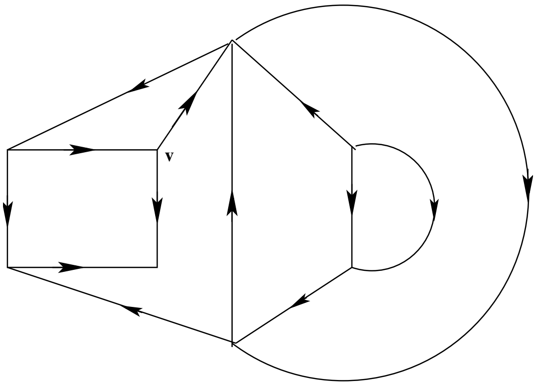

Figure 1: Example of an oriented metric graph.

length and denote by $\{E_i, i \in I\}$ the set of all edges on G. For any two points $x, y \in E_i$, we denote by $d(x, y)$ the distance between x and y induced by the isometry. We assume that G is arc connected and for any $x, y \in G$, we denote by $d(x, y)$ the natural metric on G obtained as the minimal length of all paths from x to y.

To each edge E_i, we associate an isometry $e_i : J_i \to \bar{E}_i$, with $J_i = [0, L_i]$ when $L_i < \infty$ and $J_i = [0, \infty)$ or $J_i = (-\infty, 0]$ when $L_i = \infty$. Note that $e_i(t) \in E_i$ for all t in the interior of J_i, $e_i(0) \in V$ and when $L_i < \infty$, $e_i(L_i) \in V$. The mapping e_i will be called the orientation of the edge E_i and the family $\mathcal{E} = \{e_i;\ i \in I\}$ defines the orientation of G. When $L_i < \infty$, set $\{g_i, d_i\} = \{e_i(0), e_i(L_i)\}$. When $J_i = [0, \infty)$, set $\{g_i, d_i\} = \{e_i(0), \infty\}$ and when $J_i = (-\infty, 0]$ set $\{g_i, d_i\} = \{\infty, e_i(0)\}$. For all $v \in V$, set $I_v^+ = \{i \in I;\ g_i = v\}$, $I_v^- = \{i \in I;\ d_i = v\}$ and $I_v = I_v^+ \cup I_v^-$. Let n_v, n_v^+ and n_v^- denote respectively the numbers of elements in I_v, I_v^+ and I_v^-. Then $n_v = n_v^+ + n_v^-$.

We will always assume that

- $n_v < \infty$ for all $v \in V$ (i.e. I_v is a finite set);
- $\inf_i L_i = L > 0$.

A graph with only one vertex and such that $L_i = \infty$ for all $i \in I$ will be called a star graph. It will also be convenient to imbed any star graph in the complex plane $\mathbb{C}$. Its unique vertex will be denoted by 0.

For each $v \in V$, define $G_v = \{v\} \cup \cup_{i \in I_v} E_i$ and $G_v^L = \{x \in G;\ d(x, v) < L\}$, which is then a subset of G_v. Note that $G_v \cap V = \{v\}$. For each $v \in V$, there exists a star graph $\hat{G}_v$ and a mapping $i_v : G_v \to \hat{G}_v$ such that $i_v : G_v \to i_v(G_v)$ is an isometry. This implies in particular that $i_v(v) = 0$ and that $\hat{G}_v^L = \{x \in \hat{G}_v;\ d(0, v) < L\} = i_v(G_v^L)$. For each $i \in I_v$, define $\hat{e}_i^v = i_v \circ e_i$. Note that $\hat{G}_v$ can be written in the form $\{0\} \cup \cup_{i \in I_v} \hat{E}_i^v$, with $i_v(E_i) \subset \hat{E}_i^v$ and where $(\hat{E}_i^v)_{i \in I_v}$ is the set of edges of $\hat{G}^v$. The mapping $\hat{e}_i^v$ can be extended to an isometry $(-\infty, 0] \to \{0\} \cup \hat{E}_i^v$ when $i \in I_v^-$ and to an isometry $[0, +\infty) \to \{0\} \cup \hat{E}_i^v$ when $i \in I_v^+$.

For $x \in G_v$ and $f : G_v \to \mathbb{R}$, set $\hat{x}_v := i_v(x)$ and let $\hat{f}_v : \hat{G}_v \to \mathbb{R}$ be the mapping defined by $\hat{f} = 0$ on $i_v(G_v)^c$ and $\hat{f}_v = f \circ i_v^{-1}$ on $i_v(G_v)$, so that $\hat{f}_v(\hat{x}_v) = f(x)$ for all $x \in G_v$.

We will also denote by $\mathcal{B}(G)$ the set of Borel sets of G and by $\mathcal{P}(G)$ the set of Borel probability measures on G. Recall that a kernel on G is a measurable mapping $k : G \to \mathcal{P}(G)$. For $x \in G$ and $A \in \mathcal{B}(G)$, $k(x, A)$ denotes $k(x)(A)$ and the probability measure $k(x)$ will sometimes be denoted by $k(x, dy)$. For f a bounded measurable mapping on

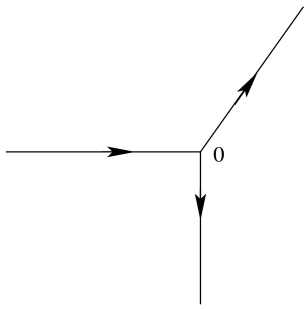

Figure 2: The star graph $\hat{G}_v$ associated to v in Figure 1.

G, $kf(x)$ denotes $\int f(y)k(x,dy)$. The set of all continuous functions on G which vanish at infinity will be denoted by $C_0(G)$.

2.2 SDE on G

Let G be an oriented metric graph. To each $v \in V$ and $i \in I_v$, we associate a transmission parameter $\alpha_v^i > 0$ such that $\sum_{i\in I_v} \alpha_v^i = 1$ and set $\alpha = (\alpha_v^i;\ v \in V,\ i \in I_v)$. Define $\mathcal{D}_\alpha^G$ the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in I$, $f \circ e_i$ is C^2 on the interior of J_i with bounded first and second derivatives both extendable by continuity to J_i and such that for all $v \in V$

$$\begin{aligned}\sum_{i\in I_v^+} \alpha_v^i \lim_{r\to 0+} (f\circ e_i)'(r) &= \sum_{i\in I_v^-, L_i<\infty} \alpha_v^i \lim_{r\to L_i-} (f\circ e_i)'(r) \\ &+ \sum_{i\in I_v^-, L_i=\infty} \alpha_v^i \lim_{r\to 0-} (f\circ e_i)'(r).\end{aligned}$$

Since α will be fixed, $\mathcal{D}_\alpha^G$ will simply be denoted by $\mathcal{D}$. When $\hat{G}_v$ is a star graph as defined before, to the half line $\hat{E}_i^v$, we associate the parameter α_v^i. Set $\alpha_v = (\alpha_v^i;\ i \in I_v)$ and $\hat{\mathcal{D}}_v = \mathcal{D}_{\alpha_v}^{\hat{G}_v}$. For $f \in \mathcal{D}$ and $x = e_i(r) \in G\backslash V$, set $f'(x) = (f\circ e_i)'(r)$, $f''(x) = (f\circ e_i)''(r)$ and take the convention $f'(v) = f''(v) = 0$ for all $v \in V$.

Definition 2.1. *A stochastic flow of kernels (SFK) K on G, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, is a family $(K_{s,t})_{s\le t}$ such that*

1. *For all $s \le t$, $K_{s,t}$ is a measurable mapping from $(G\times\Omega, \mathcal{B}(G)\otimes\mathcal{A})$ to $(\mathcal{P}(G), \mathcal{B}(\mathcal{P}(G)))$;*
2. *For all $h \in \mathbb{R}$, $s \le t$, $K_{s+h,t+h}$ is distributed like $K_{s,t}$;*
3. *For all $s_1 \le t_1 \le \cdots \le s_n \le t_n$, the family $\{K_{s_i,t_i}, 1 \le i \le n\}$ is independent;*
4. *For all $s \le t \le u$ and all $x \in G$, a.s. $K_{s,u}(x) = K_{s,t}K_{t,u}(x)$, and $K_{s,s}$ equals the identity;*
5. *For all $f \in C_0(G)$, and $s \le t$, we have*

$$\lim_{(u,v)\to(s,t)} \sup_{x\in G} \mathbb{E}[(K_{u,v}f(x) - K_{s,t}f(x))^2] = 0;$$

6. *For all $f \in C_0(G)$, $x \in G$, $s \le t$, we have*

$$\lim_{y\to x} \mathbb{E}[(K_{s,t}f(y) - K_{s,t}f(x))^2] = 0;$$

7. *For all $s \le t$, $f \in C_0(G)$, $\lim_{|x|\to\infty} \mathbb{E}[(K_{s,t}f(x))^2] = 0$.*

We say that φ is a stochastic flow of mappings (SFM) on G if $K_{s,t}(x) = \delta_{\varphi_{s,t}(x)}$ is a SFK on G. Given two SFK's K^1 and K^2 on G, we say that K^1 is a modification of K^2 if for all $s \le t$, $x \in G$, a.s. $K^1_{s,t}(x) = K^2_{s,t}(x)$.

For a family of random variables $Z = (Z_{s,t})_{s\le t}$, set for all $s \le t$, $\mathcal{F}^Z_{s,t} = \sigma(Z_{u,v}, s \le u \le v \le t)$.

Definition 2.2. *(Real white noise) A family $(W_{s,t})_{s\le t}$ is called a real white noise if there exists a Brownian motion on the real line $(W_t)_{t\in\mathbb{R}}$, that is $(W_t)_{t\ge 0}$ and $(W_{-t})_{t\ge 0}$ are two independent standard Brownian motions such that for all $s \le t$, $W_{s,t} = W_t - W_s$ (in particular, when $t \ge 0$, $W_t = W_{0,t}$ and $W_{-t} = -W_{-t,0}$).*

Our main interest in this paper is the following SDE, that extends Tanaka's SDE to metric graphs.

Definition 2.3. *(Equation (E^G_α)) On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let W be a real white noise and K be a stochastic flow of kernels on G. We say that (K, W) solves (E^G_α) if for all $s \le t$, $f \in \mathcal{D}$ and $x \in G$, a.s.*

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}f'(x)W(du) + \frac{1}{2}\int_s^t K_{s,u}f''(x)du.$$

When φ is a SFM and $K = \delta_\varphi$ is a solution of (E), we simply say that (φ, W) solves (E^G_α).

Since G and α will be fixed from now on, we will denote Equation (E^G_α) simply by (E), and we will also denote $(E^{\hat{G}_v}_{\alpha_v})$ simply by $(\hat{E}^v)$. A complete classification of solutions to $(\hat{E}^v)$ has been given in [8].

A family of σ-fields $(\mathcal{F}_{s,t};\ s \le t)$ will be said *independent on disjoint time intervals (abbreviated : i.d.i)* as soon as for all $(s_i, t_i)_{1\le i\le n}$ with $s_i \le t_i \le s_{i+1}$, the σ-fields $(\mathcal{F}_{s_i,t_i})_{1\le i\le n}$ are independent. Note that for K a SFK, since the increments of K are independent, then $(\mathcal{F}^K_{s,t};\ s \le t)$ is i.d.i.

Our main result is the following

Theorem 2.4. ***(i)*** *Let W be a real white noise and let $(\hat{K}^v)_{v\in V}$ be a family of SFK's respectively on $\hat{G}_v$. Assume that, for each $v \in V$, $(\hat{K}^v, W)$ is a solution of $(\hat{E}^v)$ and that $\big(\hat{\mathcal{F}}_{s,t} := \vee_{v\in V}\mathcal{F}^{\hat{K}^v}_{s,t};\ s \le t\big)$ is independent on disjoint time intervals. Then, there exists a unique (up to modification) SFK K on G such that*

- *$\mathcal{F}^K_{s,t} \subset \hat{\mathcal{F}}_{s,t}$ for all $s \le t$,*
- *(K, W) is a solution to (E) and*
- *for all $s \in \mathbb{R}$ and $x \in G_v$, setting*

$$\rho^{x,v}_s = \inf\{u \ge s :\ K_{s,u}(x, G_v) < 1\}, \tag{2.1}$$

then for all $t > s$, a.s. on the event $\{t < \rho^{x,v}_s\}$,

$$i_v * K_{s,t}(x) = \hat{K}^v_{s,t}(\hat{x}^v). \tag{2.2}$$

(ii) *Let (K, W) be a solution of (E). Then for each $v \in V$, there exists a unique (up to modification) SFK $\hat{K}^v$ on $\hat{G}_v$ such that*

- *for all $s \le t$, $\hat{\mathcal{F}}_{s,t} := \vee_{v\in V}\mathcal{F}^{\hat{K}^v}_{s,t} \subset \mathcal{F}^K_{s,t}$,*
- *$(\hat{K}^v, W)$ is a solution of $(\hat{E}^v)$ for each $v \in V$*

and such that if $\rho^{x,v}_s$ is defined by (2.1), then for all $s < t$ in $\mathbb{R}$ and $x \in G_v$, a.s. on the event $\{t < \rho^{x,v}_s\}$, (2.2) holds.

Note that (2.2) can be rewritten : for all bounded measurable functions f on G, and all $x \in G_v$

$$K_{s,t}f(x) = \hat{K}_{s,t}\hat{f}^v(\hat{x}^v).$$

Theorem 2.4 reduces the construction of solutions to (E) to the construction of solutions to $(\hat{E}^v)$. Since for all $v \in V$, all solutions to $(\hat{E}^v)$ are described in [8], to complete the construction of all solutions to (E), one has to be able to construct them jointly. The classification is therefore complete if one wants to construct flows K solutions of (E) such that the flows $\hat{K}^v$ associated to K are independent given W.

Since for all $v \in V$, there is a unique $\sigma(W)$-measurable flow solving $(\hat{E}^v)$, Theorem 2.4 implies that there is a unique $\sigma(W)$-measurable flow solving (E). Notice also that in general, the condition $\big(\hat{\mathcal{F}}_{s,t}; s \le t\big)$ is i.d.i does not imply that the flows $\hat{K}^v$ are independent given W. And thus, even thought for all $v \in V$ there exists a unique (in law) flow of mappings solution of $(\hat{E}^v)$, it may be possible, assuming only that $\big(\hat{\mathcal{F}}_{s,t}; s \le t\big)$ is i.d.i, to construct different (in law) flows of mappings solving (E). However, there is a unique (in law) flow of mappings solution to (E) such that the associated flows of mappings solutions to $(\hat{E}^v)$ are independent given W.

For each $v \in V$, let $\alpha_v^+ = \sum_{i \in I_v^+} \alpha_v^i$ and $\beta_v = 2\alpha_v^+ - 1$. Under some condition linking β_{v_1} and β_{v_2}, the next proposition shows that $(\hat{\mathcal{F}}_{s,t};\ s \le t)$ is i.d.i if and only if $\hat{K}^{v_1}$ and $\hat{K}^{v_2}$ are independent given W.

Proposition 2.5. *Let v_1 and v_2 be two vertices in V such that $\beta_{v_2} \neq \beta_{v_1}$ and*

$$|\beta_{v_2} - \beta_{v_1}| \ge 2\beta_{v_1}\beta_{v_2}.$$

Let W be a real white noise. Let $\hat{K}^{v_1}$ and $\hat{K}^{v_2}$ be SFK's respectively on $\hat{G}^{v_1}$ and on $\hat{G}^{v_2}$ such that $(\hat{K}^{v_1}, W)$ and $(\hat{K}^{v_2}, W)$ are solutions respectively to $(\hat{E}^{v_1})$ and to $(\hat{E}^{v_2})$. Then $(\mathcal{F}_{s,t}^{\hat{K}^{v_1}} \vee \mathcal{F}_{s,t}^{\hat{K}^{v_2}})_{s \le t}$ is i.d.i if and only if $\hat{K}^{v_1}$ and $\hat{K}^{v_2}$ are independent given W.

This proposition has been proved in [10] in the case where $V = \{v_1, v_2\}$, with $\hat{G}^{v_1}$ and $\hat{G}^{v_2}$ being given by the following star graphs

Figure 3: $\hat{G}^{v_1}$ and $\hat{G}^{v_2}$.

3 Construction of a solution of (E) out of solutions of $(\hat{E}^v)$

For all $i \in I$, let W^i be a real white noise. Assume that $\mathcal{W} := (W_{s,t}^i;\ i \in I,\ s \le t)$ is Gaussian. Let

$$A_{s,t} := \{\sup_{i \in I} \sup_{s<u<v<t} |W_{u,v}^i| < L\}. \tag{3.1}$$

Assume that $\lim_{|t-s| \to 0} +P(A_{s,t}^c) = 0$. Note that this assumption is satisfied if $W^i = W$ for all i, or if I is finite.

Let $\hat{K} = (\hat{K}^v)_{v \in V}$ be a family of SFK's respectively on $\hat{G}_v$ and let $\mathcal{W}^v := (W^i;\ i \in I_v)$. Assume that $(\hat{K}^v, \mathcal{W}^v)$ is a solution to the following SDE : for all $s \le t$, $\hat{f} \in \hat{\mathcal{D}}_v$, $\hat{x} \in \hat{G}_v$, a.s.

$$\hat{K}_{s,t}^v \hat{f}(\hat{x}) = \hat{f}(\hat{x}) + \sum_{i \in I_v} \int_s^t \hat{K}_{s,u}^v(1_{\hat{E}_i^v}\hat{f}')(\hat{x})W^i(du) + \frac{1}{2}\int_s^t \hat{K}_{s,u}^v \hat{f}''(\hat{x})du. \tag{3.2}$$

Then we have the following

Lemma 3.1. *For all $v \in V$, $i \in I_v$ and all $s \le t$, we have $\mathcal{F}_{s,t}^{W^i} \subset \mathcal{F}_{s,t}^{\hat{K}^v}$.*

Proof. Let $y = \hat{e}_i^v(r) \in \hat{E}_i^v$. Following Lemma 6 [8], we prove that $\hat{K}_{s,t}^v(y) = \delta_{\hat{e}_i^v(r+W_{s,t}^i)}$ for all $s \le t \le \sigma_s^y$ where

$$\sigma_s^y = \inf\{u \ge s;\ \hat{e}_i^v(r + W_{s,u}^i) = 0\}.$$

Since this holds for arbitrarily large r, the lemma is proved. $\square$

In all this section, we assume that

$$\left(\hat{\mathcal{F}}_{s,t} := \vee_{v\in V} \mathcal{F}_{s,t}^{\hat{K}^v};\ s \le t\right) \quad \text{is i.d.i.} \tag{3.3}$$

We will prove the following

Theorem 3.2. *There exists K a unique (up to modification) SFK on G, such that*

- $\mathcal{F}_{s,t}^K \subset \hat{\mathcal{F}}_{s,t}$ *for all* $s \le t$,
- (K, W) *is a solution to the SDE : for all* $s \le t$, $f \in \mathcal{D}$ *and* $x \in G$, *a.s.*

$$K_{s,t}f(x) = f(x) + \sum_{i\in I} \int_s^t K_{s,u}(1_{E_i} f')(x) W^i(du) + \frac{1}{2}\int_s^t K_{s,u} f''(x) du$$

and such that, defining for $s \in \mathbb{R}$, $v \in V$ *and* $x \in G_v$,

$$\rho_s^{x,v} = \inf\{t \ge s :\ K_{s,t}(x, G_v) < 1\}, \tag{3.4}$$

we have that for all $s < t$ *in* $\mathbb{R}$ *and* $x \in G_v$, *a.s. on the event* $\{t < \rho_s^{x,v}\}$,

$$i_v * K_{s,t}(x) = \hat{K}_{s,t}^v(\hat{x}^v). \tag{3.5}$$

Note that Theorem 3.2 implies (i) of Theorem 2.4.

3.1 Construction of K

For all $s \in \mathbb{R}$ and $x \in G$, define

$$\tau_s^x = \inf\{t \ge s;\ e_i(r + W_{s,t}^i) \in V\}$$

where $i \in I$ and $r \in J_i$ are such that $x = e_i(r)$. For $s < t$, define the kernel $K_{s,t}^0$ on G by : on the event $A_{s,t}^c$, set $K_{s,t}^0(x) = \delta_x$, and on the event $A_{s,t}$, if $x = e_i(r) \in G$ and $v = e_i(r + W_{s,\tau_s^x}^i)$, set

$$K_{s,t}^0(x) \quad = \quad \begin{cases} \delta_{e_i(r+W_{s,t}^i)}, & \text{if} \quad t \le \tau_s^x \\ i_v^{-1} * \hat{K}_{s,t}^v(\hat{x}^v), & \text{if} \quad t > \tau_s^x \end{cases}$$

(i.e. for $A \in \mathcal{B}(G)$, $\left(i_v^{-1} * \hat{K}_{s,t}^v(\hat{x}^v)\right)(A) = \hat{K}_{s,t}^v(\hat{x}^v, i_v(A \cap G_v))$). Note that on $A_{s,t} \cap \{t > \tau_s^x\} \cap \{v = e_i(r + W_{s,\tau_s^x}^i)\}$, we have that the support of $\hat{K}_{s,t}^v(\hat{x}^v)$ is included in $i_v(G_v)$ so that $K_{s,t}^0(x) \in \mathcal{P}(G)$. Remark also that on $A_{s,t} \cap \{v = e_i(r + W_{s,\tau_s^x}^i)\}$, a.s.

$$K_{s,t}^0(x) = i_v^{-1} * \hat{K}_{s,t}^v(\hat{x}^v).$$

Lemma 3.3. *For all* $s < t < u$ *and all* $\mu \in \mathcal{P}(G)$, *a.s. on* $A_{s,u}$,

$$\mu K_{s,u}^0 = \mu K_{s,t}^0 K_{t,u}^0. \tag{3.6}$$

Proof. Fix $s < t < u$ and note that $A_{s,u} \subset A_{s,t} \cap A_{t,u}$ a.s. We will prove the lemma only for $\mu = \delta_x$ which is enough since by Fubini's Theorem : $\forall A \in \mathcal{B}(G)$

$$\mathbb{E}[|\mu K^0_{s,u}(A) - \mu K^0_{s,t}K^0_{t,u}(A)|] \leq \int_G \mathbb{E}[|K^0_{s,u}(x,A) - K^0_{s,t}K^0_{t,u}(x,A)|]\mu(dx).$$

Let i and r be such that $x = e_i(r)$.

When $t \leq \tau^x_s$, set $Y = e_i(r + W^i_{s,t})$. If $u \leq \tau^x_s$, then it is easy to see that (3.6) holds after having remarked that $\tau^Y_t = \tau^x_s$. If $t \leq \tau^x_s < u$, then $K^0_{s,t}(x) = \delta_Y$ and $K^0_{s,u}(x) = i_v^{-1} * \hat{K}^v_{s,u}(\hat{x}^v)$ with $v = e_i(r + W^i_{s,\tau^x_s})$. We still have $\tau^Y_t = \tau^x_s$ which is now less than u. Write $K^0_{s,t}K^0_{t,u}(x) = K^0_{t,u}(Y) = i_{v'}^{-1} * \hat{K}^{v'}_{t,u}(\hat{Y}^v)$ where $v' = e_i(e_i^{-1}(Y) + W^i_{t,\tau^Y_t})$. Since $e_i^{-1}(Y) = r + W^i_{s,t}$, we have

$$v' = e_i(r + W^i_{s,t} + W^i_{t,\tau^Y_t}) = e_i(r + W^i_{s,\tau^x_s}) = v.$$

Since $\hat{K}^v$ is a flow, we get

$$K^0_{s,t}K^0_{t,u}(x) = i_v^{-1} * \hat{K}^v_{t,u}(\hat{Y}^v) = i_v^{-1} * \hat{K}^v_{s,t}\hat{K}^v_{t,u}(\hat{x}^v) = i_v^{-1} * \hat{K}^v_{s,u}(\hat{x}^v) = K^0_{s,u}(x).$$

When $t > \tau^x_s$, then $K^0_{s,t}(x) = i_v^{-1} * \hat{K}^v_{s,t}(\hat{x}^v)$ and $K^0_{s,u}(x) = i_v^{-1} * \hat{K}^v_{s,u}(\hat{x}^v)$ with v defined as above. Let f be a bounded measurable function on G. Then $K^0_{s,u}f(x) = \hat{K}^v_{s,u}\hat{f}^v(\hat{x}^v)$. Since $\hat{K}^v$ is a flow, we have

$$K^0_{s,u}f(x) = \hat{K}^v_{s,t}\hat{K}^v_{t,u}\hat{f}^v(\hat{x}^v).$$

Note that on the event $A_{s,t} \cap \{\tau^x_s < t\}$, the support of $K^0_{s,t}(x)$ is included in G^L_v, and for all y in the support of $K^0_{s,t}(x)$, the support of $\hat{K}^v_{t,u}(\hat{y}^v)$ is included in $\hat{G}^L_v$. In other words, it holds that on the event $A_{s,t} \cap \{\tau^x_s < t\}$, for all y in the support of $K^0_{s,t}(x)$, $\hat{K}^v_{t,u}\hat{f}^v(\hat{y}^v) = K^0_{t,u}f(y)$ and thus that $\hat{K}^v_{s,t}\hat{K}^v_{t,u}\hat{f}^v(\hat{y}^v) = K^0_{s,t}K^0_{t,u}f(y)$. This implies the lemma. □

We will say that a random kernel K is *Fellerian* when for all $n \geq 1$ and all $h \in C_0(G^n)$, we have $\mathbb{E}[K^{\otimes n}h] \in C_0(G^n)$.

Lemma 3.4. *For all $s < t$, $K^0_{s,t}$ is Fellerian.*

Proof. By an approximation argument (see the proof of Proposition 2.1 [16]), it is enough to prove the following L^2-continuity for K^0 : for all $f \in C_0(G)$ and all $x \in G$, $\lim_{y\to x} \mathbb{E}[(K^0_{0,t}f(y) - K^0_{0,t}f(x))^2] = 0$. Write

$$(K^0_{0,t}f(y) - K^0_{0,t}f(x))^2 = (K^0_{0,t}f(y) - K^0_{0,t}f(x))^2 1_{A_{0,t}} + (f(y) - f(x))^2 1_{A^c_{0,t}}.$$

Suppose that x belongs to an edge E_i. Using the convergence in probability $W^i_{\tau^y_0} \to W^i_{\tau^x_0}$ as $y \to x$, we see that $\mathbb{P}(K^0_{0,\tau^y_0}(y) \neq K^0_{0,\tau^x_0}(x))$ converges to 0 as $y \to x$. To conclude, it remains to prove that for $v \in \{g_i, d_i\}$ (i.e. v is an end point of E_i), letting $C^v_t = A_{0,t} \cap \{K^0_{0,\tau^y_0}(y) = K^0_{0,\tau^x_0}(x) = \delta_v\}$, we have

$$\lim_{y\to x} \mathbb{E}[(K^0_{0,t}f(y) - K^0_{0,t}f(x))^2 1_{C^v_t}] = 0.$$

Since on C^v_t, $K^0_{0,t}(z) = i_v^{-1} * \hat{K}^v_{0,t}(\hat{z}^v)$ for $z \in \{x, y\}$, our result holds. □

Lemma 3.5. *Let K_1 and K_2 be two independent Fellerian kernels. Then K_1K_2 is a Fellerian kernel.*

Proof. Set $\mathsf{P}_1^{(n)} = \mathbb{E}[K_1^{\otimes n}]$ and $\mathsf{P}_2^{(n)} = \mathbb{E}[K_2^{\otimes n}]$. Then $\mathsf{P}_1^{(n)}\mathsf{P}_2^{(n)} = \mathbb{E}[(K_1K_2)^{\otimes n}]$. This implies the lemma. $\square$

Define for $n \in \mathbb{N}$, $\mathbb{D}_n := \{k2^{-n};\ k \in \mathbb{Z}\}$. For $s \in \mathbb{R}$, let $s_n = \sup\{u \in \mathbb{D}_n;\ u \le s\}$ and $s_n^+ = s_n + 2^{-n}$. For every $n \ge 1$ and $s \le t$ define

$$K_{s,t}^n = K^0_{s,s_n^+} K^0_{s_n^+,s_n^+ + 2^{-n}} \dots K^0_{t_n - 2^{-n},t_n} K^0_{t_n,t},$$

if $s_n^+ \le t$ and $K_{s,t}^n = K_{s,t}^0$, if $s_n^+ > t$. Note that Lemma 3.4 and Lemma 3.5 imply that $K_{s,t}^n$ is Fellerian (since the kernels K^0_{s,s_n^+}, $K^0_{s_n^+,s_n^+ + 2^{-n}}$, $\dots, K^0_{t_n - 2^{-n},t_n}$, $K^0_{t_n,t}$ are independent by (3.3)).

Define $\Omega_{s,t}^n = \{\sup_i \sup_{\{s<u<v<t;\, |v-u|\le 2^{-n}\}} |W_{u,v}^i| < L\}$. Note that for all $s \le u < v \le t$ such that $|u - v| \le 2^{-n}$, we have $\Omega_{s,t}^n \subset A_{u,v}$. Let $\Omega_{s,t} = \cup_n \Omega_{s,t}^n$, then $\mathbb{P}(\Omega_{s,t}) = 1$. Define now, for $\omega \in \Omega_{s,t}$, $K_{s,t}(\omega) = K_{s,t}^n(\omega)$ where $n = n_{s,t} = \inf\{k;\ \omega \in \Omega_{s,t}^k\}$ and set $K_{s,t}(x) = \delta_x$ on $\Omega_{s,t}^c$.

Lemma 3.6. *For all $s < t$ and all $\mu \in \mathcal{P}(G)$, a.s. we have*

$$\mu K_{s,t}^m = \mu K_{s,t} \quad \textit{for all } m \ge n_{s,t}.$$

Proof. For $m \ge n_{s,t}$, we have (denoting $n = n_{s,t}$)

$$\mu K_{s,t} = \mu K^0_{s,s_n^+} K^0_{s_n^+,s_n^+ + 2^{-n}} \dots K^0_{t_n - 2^{-n},t_n} K^0_{t_n,t}.$$

where $s_n^+, s_n^+ + 2^{-n}, \cdots, t_n$ are also in $\mathbb{D}_m$. Moreover, for all $(u,v) \in \{(s,s_n^+), (s_n^+, s_n^+ + 2^{-n}), \cdots, (t_n,t)\}$, we have $\Omega_{s,t}^n \subset A_{u,v}$. Now applying Lemma 3.3 and an independence argument, we see that $\mu K_{s,t} = \mu K_{s,t}^m$. $\square$

Proposition 3.7. *K is a SFK.*

Proof. Obviously the increments of K are independent. Fix $s < t < u$, then by the previous lemma and Lemma 3.3 a.s. for m large enough (i.e. $m \ge \max\{n_{s,u}, n_{s,t}, n_{t,u}\}$), we have

$$\begin{aligned}
\mu K_{s,u} &= \mu K_{s,u}^m = \mu K_{s,t_m}^m K_{t_m,t_m^+}^m K_{t_m^+,t_m^+ + 2^{-m}}^m \cdots K_{u_m,u}^m \\
&= \mu K_{s,t_m}^m K_{t_m,t}^m K_{t,t_m^+}^m K_{t_m^+,t_m^+ + 2^{-m}}^m \cdots K_{u_m,u}^m \\
&= \mu K_{s,t}^m K_{t,u}^m \\
&= \mu K_{s,t} K_{t,u}.
\end{aligned}$$

This proves that K satisfies the flow property.

Fix $k \ge 1$, $h \in C^0(G^k)$. Let $\epsilon > 0$ and $n_1 \in \mathbb{N}$ such that $\mathbb{P}(n_{s,t} > n_1) < \epsilon$. Then for all $(x,y) \in G^k \times G^k$, since $\mathbb{P}(\Omega_{s,t}) = 1$, we have

$$\begin{aligned}
|\mathbb{E}[K_{s,t}^{\otimes k} h(y)] - \mathbb{E}[K_{s,t}^{\otimes k} h(x)]| &\le \sum_{n \le n_1} \mathbb{E}\big[\big((K_{s,t}^n)^{\otimes k} h(y) - (K_{s,t}^n)^{\otimes k} h(x)\big)^2\big]^{\frac{1}{2}} \\
&+ 2\epsilon \|h\|_\infty.
\end{aligned}$$

Now since $K_{s,t}^n$ is Fellerian for all n, we deduce that

$$\limsup_{y \to x} |\mathbb{E}[K_{s,t}^{\otimes k} h(y)] - \mathbb{E}[K_{s,t}^{\otimes k} h(x)]| \le 2\epsilon \|h\|_\infty.$$

Since ϵ is arbitrary, it holds that for all $s < t$, $K_{s,t}$ is Fellerian.

Lemma 3.8. *For all $x \in G$ and $f \in C_0(G)$, $\lim_{|t-s|\to 0} \mathbb{E}[(K_{s,t}f(x) - f(x))^2] = 0$.*

Proof. Take $x = e_i(r)$ and let $\epsilon > 0$. Then, there exists $\alpha > 0$ such that $|t-s| < \alpha$ implies $\mathbb{P}(A_{s,t}) > 1-\epsilon$. Note that a.s. on $A_{s,t}$, $K_{s,t}(x) = K^0_{s,t}(x)$. If $x \notin V$, then $\mathbb{E}[(K_{s,t}f(x) - f(x))^2 1_{A_{s,t}}] \le 2\|f\|^2_\infty \mathbb{P}(\tau^x_s < t) + \mathbb{E}[(f(e_i(r + W^i_{s,t})) - f(e(r)))^2 1_{t \le \tau^x_s}]$. The two right hand terms clearly converge to 0 as $|t-s|$ goes to 0. This implies the lemma when $x \notin V$. When $x = v \in V$, then a.s. on $A_{s,t}$, $K_{s,t}f(x) = \hat{K}^v_{s,t}\hat{f}^v(0)$. We can conclude the proof now since $\hat{K}^v$ is a SFK. □

Lemma 3.8 together with the flow property implies that for all $f \in C_0(G)$ and all $x \in G$, $(s,t) \mapsto K_{s,t}f(x)$ is continuous as a mapping from $\{s < t\} \to L^2(\mathbb{P})$. Now since for all $s < t$ in $\mathbb{D}$, the law of $K_{s,t}$ only depends on $|t-s|$, the continuity of this mapping implies that this also holds for all $s < t$. Thus, we have proved that K is a SFK. □

3.2 The SDE satisfied by K

Recall that each flow $\hat{K}^v$ solves equation $(\hat{E}^v)$ defined on $\hat{G}_v$. Then we have

Lemma 3.9. *For all $x \in G$, $f \in \mathcal{D}$ and all $s < t$, a.s. on $A_{s,t}$*

$$K^0_{s,t}f(x) = f(x) + \sum_{i \in I} \int_s^t K^0_{s,u}(1_{E_i} f')(x) W^i(du) + \frac{1}{2}\int_s^t K^0_{s,u} f''(x) du.$$

Proof. Let $x = e_i(r)$ with $i \in I_v$. Recall the notation $\hat{x}_v = i_v(x) \in \hat{G}_v$. Then denoting $B^v_{s,t} = A_{s,t} \cap \{\tau^x_s \le t\} \cap \{e_i(r + W^i_{s,\tau^x_s}) = v\}$, we have that a.s. on $A_{s,t}$,

$$K^0_{s,t}f(x) \quad = \quad (f \circ e_i)(r + W^i_{s,t}) 1_{\{\tau^x_s > t\}} + \sum_{v \in V} \hat{K}^v_{s,t} \hat{f}^v(\hat{x}_v) 1_{B^v_{s,t}}.$$

Thus a.s. on $A_{s,t}$,

$$\begin{aligned} K^0_{s,t}f(x) &= f(x) \\ &+ 1_{\{\tau^x_s > t\}} \left(\int_s^t (f \circ e_i)'(r + W^i_{s,u}) W^i(du) + \frac{1}{2}\int_s^t (f \circ e_i)''(r + W^i_{s,u}) du \right) \\ &+ \sum_{v \in V} 1_{B^v_{s,t}} \left(\sum_{j \in I_v} \int_s^t \hat{K}^v_{s,u}\big(1_{\hat{E}^v_j} (\hat{f}^v)'\big)(\hat{x}_v) W^j(du) + \int_s^t \hat{K}^v_{s,u} (\hat{f}^v)''(\hat{x}_v) du \right) \\ &= f(x) + 1_{\{\tau^x_s > t\}} \left(\int_s^t K^0_{s,u}(1_{E_i} f')(x) W^i(du) + \frac{1}{2}\int_s^t K^0_{s,u} f''(x) du \right) \\ &\quad + \sum_{v \in V} 1_{B^v_{s,t}} \left(\sum_{j \in I_v} \int_s^t K^0_{s,u}(1_{E_j} f')(x) W^j(du) + \int_s^t K^0_{s,u} f''(x) du \right). \end{aligned}$$

This implies the lemma. □

Lemma 3.10. *For all $n \in \mathbb{N}$, $x \in G$, $s < t$ and all $f \in \mathcal{D}$ a.s. on $\Omega^n_{s,t}$, we have*

$$\begin{aligned} K^n_{s,t}f(x) &= f(x) + \sum_{i \in I} \int_s^t K^n_{s,u}(1_{E_i} f')(x) W^i(du) \\ &\quad + \frac{1}{2}\int_s^t K^n_{s,u} f''(x) du. \end{aligned}$$

Proof. We proceed by induction on $q = \text{Card}\,\{s, s_n^+, s_n^+ + 2^{-n}, \cdots, t_n, t\}$. For $q = 2$, this is immediate from Lemma 3.9 since $\Omega_{s,t}^n \subset A_{s,t}$. Assume this is true for $q-1$ and let $s < t$ such that $\text{Card}\,\{s, s_n^+, s_n^+ + 2^{-n}, \cdots, t_n, t\} = q$. Then a.s.

$$\begin{aligned}
K_{s,t}^n f(x) &= K_{s,t_n}^n K_{t_n,t}^n f(x) \\
&= K_{s,t_n}^n \left(f + \sum_{i \in I} \int_{t_n}^t K_{t_n,u}^n (1_{E_i} f') W^i(du) + \frac{1}{2} \int_{t_n}^t K_{t_n,u}^n f'' du \right)(x) \\
&= K_{s,t_n}^n f(x) \\
&+ \sum_{i \in I} \int_{t_n}^t K_{s,t_n}^n K_{t_n,u}^n (1_{E_i} f')(x) W^i(du) + \frac{1}{2} \int_{t_n}^t K_{s,t_n}^n K_{t_n,u}^n f''(x) du \\
&= f(x) + \sum_i \int_s^t K_{s,u}^n (1_{E_i} f')(x) W^i(du) + \frac{1}{2} \int_s^t K_{s,u}^n f''(x) du,
\end{aligned}$$

by independence of increments and using the fact that $K_{s,t_n}^n(x)$ is supported by a finite number of points. □

Thus we have

Lemma 3.11. *For all $x \in G$, $f \in \mathcal{D}$ and all $s < t$, a.s.*

$$K_{s,t} f(x) = f(x) + \sum_{i \in I} \int_s^t K_{s,u}(1_{E_i} f')(x) W^i(du) + \frac{1}{2} \int_s^t K_{s,u} f''(x) du. \tag{3.7}$$

Proof. Note that for all n, on $\Omega_{s,t}^n$, for all $u \in [s,t]$, a.s. $K_{s,u}(x) = K_{s,u}^n(x)$. Thus a.s. on $\Omega_{s,t}^n$, (3.7) holds in $L^2(\mathbb{P})$ and finally a.s. (3.7) holds. □

Remark : When $W^i = W$ for all i, then (K, W) solves the SDE (E).

Lemma 3.11 with the fact that K is a SFK permits to prove that K satisfies the first two conditions of Theorem 3.2. Note that for all $s \le t$ and all $x \in G$, we have that a.s. on $A_{s,t}$, $K_{s,t}(x) = K_{s,t}^0(x)$. Thus a.s., on $A_{s,t}$, (3.5) holds. Now, we want to prove that a.s. (3.5) holds on the event $\{t < \rho_s^{x,v}\}$ (note that a.s. $A_{s,t} \cap \{t > \tau_s^x\} \subset \{\tau_s^x < t < \rho_s^{x,v}\}$). By Lemma 3.6, a.s. for all $m \ge n_{s,t}$ such that $s_m^+ \le t$,

$$K_{s,t}(x) = K_{s,s_m^+}^0 K_{s_m^+, s_m^+ + 2^{-m}}^0 \cdots K_{t_m - 2^{-m}, t_m}^0 K_{t_m,t}^0(x).$$

Clearly on $\{t < \rho_s^{s,v}\}$, a.s. $K_{s,s_m^+}^0(x) = i_v^{-1} * \hat{K}_{s,s_m^+}^v(\hat{x}^v)$ and for all y in the support of $K_{s,s_m^+}^0(x)$, $K_{s_m^+, s_m^+ + 2^{-m}}^0(y) = i_v^{-1} * \hat{K}_{s_m^+, s_m^+ + 2^{-m}}^v(\hat{y}^v)$. Thus, on $\{t < \rho_s^{s,v}\}$, a.s.

$$K_{s,s_m^+}^0 K_{s_m^+, s_m^+ + 2^{-m}}^0(x) = i_v^{-1} * \hat{K}_{s,s_m^+}^v \hat{K}_{s_m^+, s_m^+ + 2^{-m}}^v(\hat{x}^v).$$

The same argument shows that on $\{t < \rho_s^{s,v}\}$, a.s.

$$K_{s,t}(x) = i_v^{-1} * \hat{K}_{s,s_m^+}^v \hat{K}_{s_m^+, s_m^+ + 2^{-m}}^v \cdots \hat{K}_{t_m - 2^{-m}, t_m}^v \hat{K}_{t_m,t}^v(\hat{x}^v) = i_v^{-1} * \hat{K}_{s,t}^v(\hat{x}^v).$$

To conclude the proof of Theorem 3.2, it remains to prove that if K' is a SFK satisfying also the conditions of Theorem 3.2, then K' is a modification of K. Since (3.5) holds for K and K', for all $s \le t$ and all $\mu \in \mathcal{P}(G)$ a.s. on $A_{s,t}$, $\mu K'_{s,t} = \mu K_{s,t} (= \mu K_{s,t}^0)$. Thus for all $s \le t$ and $x \in G$, denoting $n = n_{s,t}$, a.s.

$$\begin{aligned}
K'_{s,t}(x) &= K'_{s,s_n^+} \cdots K'_{t_n,t}(x) \\
&= K_{s,s_n^+} \cdots K_{t_n,t}(x) \\
&= K_{s,t}(x).
\end{aligned}$$

4 Construction of solutions of $(\hat{E}^v)$ out of a solution of (E).

Let $\mathcal{W} = (W^i;\ i \in I)$ be as in the previous section. Let K be a SFK on G. Assume that (K, W) satisfies the SDE : for all $s \le t$, $f \in \mathcal{D}$, $x \in G$, a.s.

$$K_{s,t}f(x) = f(x) + \sum_{i\in I} \int_s^t K_{s,u}(1_{E_i} f')(x) W^i(du) + \frac{1}{2}\int_s^t K_{s,u} f''(x) du.$$

Following [10, Lemma 3], one can prove that $\mathcal{F}^{W^i}_{s,t} \subset \mathcal{F}^K_{s,t}$ for all $i \in I$ and $s \le t$. In this section, we will prove the following

Theorem 4.1. *For each $v \in V$, there exists a unique (up to modification) SFK $\hat{K}^v$ on $\hat{G}^v$ such that*

- *for all $s \le t$, $\hat{\mathcal{F}}_{s,t} := \vee_{v\in V} \mathcal{F}^{\hat{K}^v}_{s,t} \subset \mathcal{F}^K_{s,t}$,*
- *for all $v \in V$, $(\hat{K}^v, \mathcal{W}^v)$ is a solution to the SDE : for all $s \le t$, $\hat{f} \in \hat{\mathcal{D}}_v$, $\hat{x} \in \hat{G}_v$, a.s.*

$$\hat{K}^v_{s,t}\hat{f}(\hat{x}) = \hat{f}(\hat{x}) + \sum_{i\in I_v} \int_s^t \hat{K}^v_{s,u}(1_{\hat{E}^v_i} \hat{f}')(\hat{x}) W^i(du) + \frac{1}{2}\int_s^t \hat{K}^v_{s,u} \hat{f}''(\hat{x}) du. \qquad (4.1)$$

and such that defining for $s \in \mathbb{R}$ and $x \in G_v$, $\rho^{x,v}_s$ by (3.4), we have that for all $t > s$, a.s. on the event $\{t < \rho^{x,v}_s\}$, (3.5) holds.

Proof. Fix $v \in V$. For $s \in \mathbb{R}$ and $\hat{x} = \hat{e}^v_i(r) \in \hat{G}_v$ with $\hat{x} \in i_v(G_v)$, set $x = e_i(r)$ (and we have $\hat{x} = i_v(x)$). Recall the definition of τ^x_s. Recall also the definition of $A_{s,t}$ from (3.1). Define the kernel $\hat{K}^{0,v}_{s,t}$ by :

- On $A^c_{s,t}$: $\hat{K}^{0,v}_{s,t}(\hat{x}) = \delta_{\hat{x}}$;
- On $A_{s,t}$: Let $\hat{x} = \hat{e}^v_i(r)$. If $\hat{x} = \hat{e}^v_i(r) \in i_v(G_v)$, $\tau^x_s < t$ and $e_i(r + W^i_{s,\tau^x_s}) = v$, define $\hat{K}^{0,v}_{s,t}(\hat{x}) = i_v * K_{s,t}(e_i(r))$. And otherwise, define $\hat{K}^{0,v}_{s,t}(\hat{x}) = \delta_{\hat{e}^v_i(r+W^i_{s,t})}$.

Now for $n \ge 1$, set

$$\hat{K}^{n,v}_{s,t} = \hat{K}^{0,v}_{s,s^+_n} \hat{K}^{0,v}_{s^+_n, s^+_n+2^{-n}} \dots \hat{K}^{0,v}_{t_n-2^{-n},t_n} \hat{K}^{0,v}_{t_n,t},$$

if $s^+_n \le t$ and $\hat{K}^{n,v}_{s,t} = \hat{K}^{0,v}_{s,t}$, if $s^+_n > t$.
Define $\Omega^n_{s,t}$, $\Omega_{s,t}$ and $n_{s,t}$ as in Section 3.1 and finally set $\hat{K}^v_{s,t} = \hat{K}^{n,v}_{s,t}$, where $n = n_{s,t}$ and $\hat{K}^v_{s,t}(\hat{x}) = \delta_{\hat{x}}$ on $\Omega^c_{s,t}$. Following Sections 3.1 and 3.2, we prove that $\hat{K}^v$ is a SFK satisfying (4.1). Note that for all $s \le t$, $x \in G_v$, $K_{s,t}(x) = i_v^{-1} * \hat{K}^{0,v}_{s,t}(\hat{x}^v)$. Since for all $s \le t$ and $\hat{x} \in \hat{G}_v$, a.s. on $A_{s,t}$, $\hat{K}^v_{s,t} = \hat{K}^{0,v}_{s,t}$, the last statement of the theorem holds. It remains to remark the uniqueness up to modification, which can be proved in the same manner as for Theorem 3.2. ☐

The previous theorem implies (ii) of Theorem 2.4.

5 Stochastic flows on star graphs [8].

In this section, we overview the content of [8] where equation (E^G_α) has been studied when G is a star graph. Let $G = \{0\} \cup \cup_{i\in I} E_i$ be a star graph where $I = \{1, \cdots, n\}$. Assume that $I_+ = \{i : g_i = 0\} = \{1, \cdots, n^+\}$ and $I_- = \{i : d_i = 0\} = \{n^+ + 1, \cdots, n\}$ and set $n^- = n - n^+$. To each edge E_i, we associate $\alpha^i \in [0, 1]$ such that $\sum_{i\in I} \alpha^i = 1$. Denote by e_i the orientation of E_i and let $\alpha^+ = \sum_{i\in I_+} \alpha^i$, $\alpha^- = 1-\alpha^+$. Let $\alpha = (\alpha^i)_{i\in I}$. In this section, we denote (E^G_α) simply by (E).

The construction of flows associated to (E) is based on the skew Brownian motion (SBM) flow studied by Burdzy and Kaspi in [3]. Let W be a real white noise, then the Burdzy-Kaspi (BK) flow Y associated to W and $\beta \in [-1,1]$ is a SFM (see Section 7 for the definition) solution to

$$Y_{s,t}(x) = x + W_{s,t} + \beta L_{s,t}(x), \tag{5.1}$$

where $L_{s,t}(x)$ is the local time of $Y_{s,\cdot}(x)$ at time t. For $x \in G$, $i \in I$ and $r \in \mathbb{R}$ such that $x = e_i(r)$, define

$$\tau_s^x = \inf\{t \geq s : e_i(r + W_{s,t}) = 0\}.$$

5.1 The case $\alpha^+ \neq \frac{1}{2}$.

For $k \geq 1$, let $\Delta_k = \left\{u \in [0,1]^k : \sum_{i=1}^k u_i = 1\right\}$. From [8], we recall the following

Theorem 5.1. *Let m^+ and m^- be two probability measures respectively on Δ_{n^+} and Δ_{n^-} satisfying : $\forall i \in [1, n^+]$ and $j \in [1, n^-]$,*

$$(+) \int_{\Delta_{n^+}} u_i m^+(du) = \frac{\alpha^i}{\alpha^+}, \quad (-) \int_{\Delta_{n^-}} u_j m^-(du) = \frac{\alpha^{j+n^+}}{\alpha^-}.$$

(a) There exists a solution (K, W) on G unique in law such that if Y is the BK flow associated to W and $\beta = 2\alpha^+ - 1$, then for all $s \leq t$ in $\mathbb{R}$, $x \in G$ a.s.

- ***(i)*** *If $x = e_i(r)$, then $K_{s,t}(x) = \delta_{e_i(r+W_{s,t})}$ on $\{t \leq \tau_s^x\}$.*
- ***(ii)*** *On $\{t > \tau_s^x\}$, $K_{s,t}(x)$ is supported on $\{e_i(Y_{s,t}(r)), i \in I_+\}$ if $Y_{s,t}(r) > 0$ and on $\{e_i(Y_{s,t}(r)), i \in I_-\}$ if $Y_{s,t}(r) \leq 0$.*
- ***(iii)*** *On $\{t > \tau_s^x,\ \pm Y_{s,t}(r) > 0\}$, $U^\pm_{s,t}(x) = (K_{s,t}(x, E_i), i \in I_\pm)$ is independent of W and has for law $m^\pm$.*

(b) For all SFK K such that (K, W) solves (E), there exist two probability measures m^+ and m^- respectively on Δ_{n^+} and Δ_{n^-} satisfying conditions $(+)$ and $(-)$ and such that (i), (ii), (iii) above are satisfied.

Let $U^+ = (U^+(i), i \in I_+)$ and $U^- = (U^-(j), j \in I_-)$ be two random variables with values in Δ_{n^+} and Δ_{n^-} such that for each $(i,j) \in I_+ \times I_-$

$$\mathbb{P}(U^+(i) = 1) = \frac{\alpha^i}{\alpha^+}, \quad \mathbb{P}(U^-(j) = 1) = \frac{\alpha^j}{\alpha^-}.$$

Note that all coordinates of $U^\pm$ are equal to 0 expect for one coordinate which is equal therefore to 1. With m^+ and m^- being respectively the laws of U^+ and U^-, $K_{s,t}(x) = \delta_{\varphi_{s,t}(x)}$ where φ is a SFM. The flow φ is also the law unique SFM solving (E).

To $U^+ = (\frac{\alpha^i}{\alpha^+}, i \in I_+)$ and $U^- = (\frac{\alpha^j}{\alpha^-}, j \in I_-)$, is associated in the same way a Wiener i.e. $\sigma(W)$-measurable solution K^W of (E) which is also the unique (up to modification) Wiener solution to (E).

5.2 The case $\alpha^+ = \frac{1}{2}$.

In this case (E) admits only one solution K^W which is Wiener, no other solutions can be constructed by adding randomness to W. The expression of K^W is the same as the general case with $Y_{s,t}(x)$ replaced by $x + W_{s,t}$.

6 Conditional independence : proof of Proposition 2.5.

In this section, we assume that for all $i \in I$, $W^i = W$ for some real white noise W. Our purpose is to establish Proposition 2.5 already proved in [10] in a very particular case. The main idea was the following : let (φ^+, W) and $(\varphi^-, -W)$ be two SFM's solutions to Tanaka's equation

$$\varphi^{\pm}_{s,t}(x) = x \pm \int_s^t \text{sgn}(\varphi^{\pm}_{s,u}(x))dW_u.$$

We know that the laws of (φ^+, W) and (φ^-, W) are unique [17]. Let $\varphi = (\varphi^+, \varphi^-)$, then if $(\mathcal{F}^{\varphi}_{s,t})_{s\le t}$ is i.d.i, the law of φ is unique. An intuitive explanation for this is that $t \mapsto |\varphi^+_{0,t}(0)| = W_t - \inf_{0\le u\le t} W_u$ and $t \mapsto |\varphi^-_{0,t}(0)| = \sup_{0\le u\le t} W_u - W_t$ do not have common zeros after 0 so that $\text{sgn}(\varphi^+_{0,t}(0))$ should be independent of $\text{sgn}(\varphi^-_{0,t}(0))$. In the general situation, the previous reflecting Brownian motions are replaced by two SBM's associated to W and distinct skew parameters.

The proof of Proposition 2.5 will strongly rely on the following lemma.

Lemma 6.1. *Let $(\beta_1, \beta_2) \in [-1,1]^2$ with $\beta_1 \neq \beta_2$ and $|\beta_2 - \beta_1| \ge 2\beta_1\beta_2$. Let $x, y \in \mathbb{R}$ and let X, Y be solutions of*

$$X_t = x + W_t + \beta_1 L_t(X) \quad \textit{and} \quad Y_t = y + W_t + \beta_2 L_t(Y)$$

where $L_t(X)$ and $L_t(Y)$ denote the symmetric local times at 0 of X and Y. If $x \neq y$ or if $x = y = 0$, then a.s. for all $t > 0$, $X_t \neq Y_t$.

Proof. Assume first that $x = y = 0$. It is straightforward to see that the lemma holds when $\beta_1 \le 0 \le \beta_2$. The other cases follow from [2, Theorem 1.4 (i)-(ii)].

Assume now that $x \neq y$ and set $T = \inf\{t > 0;\ X_t = Y_t = 0\}$. Then necessarily, if $T < \infty$, we have $X_T = Y_T = 0$. So we can conclude using the strong Markov property at time T. □

Proof of Proposition 2.5. To simplify the notation, for $i \in \{1,2\}$, G^i, β_i, I^i, $I^{i,\pm}$ and K^i will denote respectively $\hat{G}^{v_i}$, β_{v_i}, I_{v_i}, $I^{\pm}_{v_i}$ and $\hat{K}^{v_i}$. We will also denote the edges of G^i by $(e^i_j)_{j\in I^i}$ and set $\mathcal{F}_{s,t} = \mathcal{F}^{K^1}_{s,t} \vee \mathcal{F}^{K^2}_{s,t}$ for all $s \le t$.

It is easy to see that if K^1 and K^2 are independent given W, then $(\mathcal{F}_{s,t})_{s\le t}$ is i.d.i.

Assume now that $(\mathcal{F}_{s,t})_{s\le t}$ is i.d.i. For $i \in \{1,2\}$, let Y^i be the BK flow associated to W and β_i. Using the flow property, the stationarity of the flows and the fact that $(\mathcal{F}_{s,t})_{s\le t}$ is i.d.i., we only need to prove that for all $t > 0$,

$$K^1_{0,t} \ \text{ and } \ K^2_{0,t} \text{ are independent given } W. \tag{6.1}$$

For $n \ge 1$ and $i \in \{1,2\}$, let $(x^i_j = e^i_{k^i_j}(r^i_j), 1 \le j \le n)$ be n points in G^i, where $k^i_j \in I^i$ and $r^i_j \in \mathbb{R}$. Define

$$\tau^i_j = \inf\{u \ge 0 : r^i_j + W_{0,u} = 0\}.$$

Proving (6.1) reduces to prove that

$$(K^1_{0,t}(x^1_j))_{1\le j\le n} \ \text{ and } \ (K^2_{0,t}(x^2_j))_{1\le j\le n} \text{ are independent given } W \tag{6.2}$$

for arbitrary n and (x^i_j).

Note that when $t \le \tau^i_j$, then $K^i_{0,t}(x^i_j)$ is a measurable function of W. For J^1 and J^2 two subsets of $\{1, \dots, n\}$, let

$$A_{J^1,J^2} = \left\{ t > \tau^i_j \text{ if and only if } j \in J^i \text{ for all } i = 1,2 \right\}$$

which belongs to $\sigma(W)$. Then, proving (6.2) reduces to check that (for all J^1 and J^2), given W, on A_{J^1,J^2},

$$(K^1_{0,t}(x^1_j))_{j\in J^1} \text{ and } (K^2_{0,t}(x^2_j))_{j\in J^2} \text{ are independent.} \tag{6.3}$$

For $j \in J^i$, define

$$g^i_j = \sup\{u \le t : Y^i_{0,u}(r^i_j) = 0\}.$$

Note that a.s. on A_{J^1,J^2}, by Lemma 6.1,

$$\{g^1_j :\ j \in J^1\} \cap \{g^2_j :\ j \in J^2\} = \emptyset.$$

Let $\mathcal{J} = \{J_k;\ 1 \le k \le m\}$ be a partition of $\big(\{1\}\times J^1\big) \cup \big(\{2\}\times J^2\big)$ such that for all k, we have $J_k \subset \{i\}\times J^i$ for some $i \in \{1,2\}$ and define the event

$$\begin{aligned} B_{\mathcal{J}} \quad = \quad & \Big\{g^i_j = g^{i'}_{j'} \text{ if and only if } \exists\ k \text{ such that } \big((i,j),(i',j')\big) \in J_k \times J_k\Big\} \\ & \bigcap \Big\{\forall k < k';\ \text{if } \big((i,j),(i',j')\big) \in J_k \times J_{k'} \text{ then } g^i_j < g^{i'}_{j'}\Big\} \\ & \bigcap\ A_{J^1,J^2}. \end{aligned}$$

Then a.s. $\{B_{\mathcal{J}}\}_{\mathcal{J}}$ is a partition of A_{J^1,J^2}.
For all k, choose $(i_k, j_k) \in J_k$ and let $g_k = g^{i_k}_{j_k}$. Let $u_1 < \cdots < u_{m-1}$ be fixed dyadic numbers and

$$C := C_{u_1,\ldots,u_{m-1}} = B_{\mathcal{J}} \cap \{g_k < u_k < g_{k+1};\ 1 \le k \le m-1\}.$$

Let $u = u_{m-1}$. On C, for all $(i,j) \in J_m$, $K^i_{0,t}(x^i_j)$ is $\mathcal{F}_{u,t} \vee \mathcal{F}^W_{0,u}$-measurable. Indeed : fix $(j_+, j_-) \in I^{i,+} \times I^{i,-}$ and define

$$X^{i,j}_u = \begin{cases} e^i_{j_+}(Y^i_{0,u}(r^i_j)), & \text{if } Y^i_{0,u}(r^i_j) \ge 0; \\ e^i_{j_-}(Y^v_{0,u}(r^i_j)), & \text{if } Y^i_{0,u}(r^i_j) < 0. \end{cases}$$

Then $X^{i,j}_u$ is $\mathcal{F}^W_{0,u}$-measurable,

$$\inf\{r \ge u : K^i_{u,r}(X^{i,j}_u) = \delta_0\} = \inf\{r \ge u : Y^i_{0,r}(r^i_j) = 0\} \le g_m$$

and $K^i_{0,t}(x^i_j) = K^i_{u,t}(X^{i,j}_u)$.

Moreover, on C, for all $(i,j) \in \cup_{k=1}^{m-1} J_k$, $K^i_{0,t}(x^i_j)$ is $\mathcal{F}_{0,u} \vee \mathcal{F}^W_{u,t}$-measurable (since $Y^i_{0,\cdot}(r^i_j)$ does not touch 0 in the interval $[u,t]$). Since $C \in \sigma(W)$, $\mathcal{F}_{0,u} \vee \mathcal{F}^W_{u,t}$ and $\mathcal{F}_{u,t} \vee \mathcal{F}^W_{0,u}$ are independent given W, we deduce that on C, $\big(K^i_{0,t}(x^v_j), (i,j) \in J_m\big)$ and $\big(K^i_{0,t}(x^i_j), (i,j) \in \cup_{k=1}^{m-1} J_k\big)$ are independent given W. Now an immediate induction permits to show that given W, on C, (6.3) is satisfied. Since the dyadic numbers $u_1, \cdots, u_m$ are arbitrary, we deduce that conditionally on W, on $B_{\mathcal{J}}$, (6.3) is satisfied and finally given W, on A_{J^1,J^2}, (6.3) holds. □

7 Appendix 1 : The Burdzy-Kaspi flow

In this section, we show how our construction can be simplified on some particular graphs using the BK flow [3]. Let $(W_{s,t})_{s\le t}$ be a real white noise. For $\beta = \pm 1$, the flow associated to (5.1) has a simple expression which will be referred as the BK flow. For a fixed $\beta \in]-1,1[$, Burdzy and Kaspi constructed a SFM (see 1.7 in [3]) satisfying

(i) $x \mapsto Y_{s,t}(x)$ is increasing and càdlàg for all $s \le t$ a.s.

(ii) With probability equal to 1 : for all $s \in \mathbb{R}$ and all $x \in \mathbb{R}$, $(Y_{s,t}(x), L_{s,t}(x))$ satisfies (5.1) and

$$L_{s,t}(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1_{\{|Y_{s,u}(x)| \le \varepsilon\}} du.$$

The statement (i) is a consequence of the definition of Y (see also [8, Section 3.1]) and (ii) can be found in [3, Proposition 1]. The BK flow satisfies also a strong flow property :

Proposition 7.1. *(1) Fix $x \in \mathbb{R}$ and let S be an $(\mathcal{F}^W_{-\infty,r})_{r\in\mathbb{R}}$-finite stopping time. Then $Y_{S,S+\cdot}(x)$ is the unique strong solution of the SBM equation with parameter β driven by $W_{S,S+\cdot}$. In particular $Y_{S,S+\cdot}$ is independent of $\mathcal{F}^W_{-\infty,S}$.*

(2) Let $S \le T$ be two $(\mathcal{F}^W_{-\infty,r})_{r\in\mathbb{R}}$-finite stopping times. Then a.s. for all $u \ge 0, x \in \mathbb{R}$,

$$Y_{S,T+u}(x) = Y_{T,T+u} \circ Y_{S,T}(x).$$

Proof. (1) Let $\mathcal{G}_t = \mathcal{F}^W_{-\infty,S+t}, t \ge 0$. Then $Y_{S,S+\cdot}(x)$ is $\mathcal{G}$-adapted, $W_{S+\cdot} - W_S$ is a $\mathcal{G}$-Brownian motion and by (ii) above, a.s. $\forall t \ge 0$,

$$Y_{S,S+t}(x) = x + W_{S,S+t} + \beta L_{S,S+t}(x)$$

with $L_{S,S+t}(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t 1_{\{|Y_{S,S+u}(x)| \le \varepsilon\}} du$. Now (1) follows from the main result of [11].

(2) Fix $x \in \mathbb{R}$. Then by the previous lines for all $y \in \mathbb{R}$, a.s. $\forall t \ge 0$,

$$Y_{T,T+t}(y) = y + W_{T,T+t} + \beta L_{T,T+t}(y).$$

Since $Y_{T,T+\cdot}$ is independent of $\mathcal{F}^W_{-\infty,T}$ and $Y_{S,T}$ is $\mathcal{F}^W_{-\infty,T}$ measurable (by the definition of Y), it holds that a.s. $\forall t \ge 0$,

$$Y_{T,T+t}(Y_{S,T}(x)) = Y_{S,T}(x) + W_{T,T+t} + \beta L_{T,T+t}(Y_{S,T}(x)).$$

Now, set

$$Z_t = Y_{S,t}(x) 1_{\{S \le t \le T\}} + Y_{T,t} \circ Y_{S,T}(x) 1_{\{t > T\}}.$$

Then, we easily check that a.s. $\forall t \ge S$,

$$Z_t = x + W_{S,t} + \beta \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_S^t 1_{\{|Z_u| \le \varepsilon\}} du.$$

Since the SDE (5.1) has a unique strong solution, a.s. $\forall t \ge T$, $Y_{S,t}(x) = Y_{T,t} \circ Y_{S,T}(x)$. Now using (i) above, (2) holds a.s. for all $x \in \mathbb{R}$.

□

Consider equation (E) defined on the graph G of Figure 4. We will sometimes identify G with the real line. Applying Theorem 2.4, we deduce that there exists a unique (up to modification) SFM φ on $\mathbb{R}$ such that for all $s \le t$ and all $y \in \mathbb{R}$, a.s.

$$\varphi_{s,t}(y) = y + W_{s,t} + \sum_{v \in V} (2\alpha_v^+ - 1) L^{v,y}_{s,t}(\varphi)$$

where $L^{v,y}_{s,t}(\varphi) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_s^t 1_{\{|\varphi_{s,u}(y) - v| \le \epsilon\}} du$. This flow is also a Wiener solution to the previous equation. Using the BK flow, another construction of φ is possible : to each vertex v, let us attach the BK flow Y^v associated to W, with $\beta_v := 2\alpha_v^+ - 1$. If $x = e_i(r)$, define $\varphi_{s,t}(x) = e_i(r + W_{s,t})$ until hitting a vertex point v_1 at time s_1, then "define" $\varphi_{s,t}(x)$ by $Y^{v_1}_{s_1,t}(0)$ until hitting another vertex v_2. After s_2, $\varphi_{s,t}(x)$ will be "given by" $Y^{v_2}_{s_2,t}(0)$ etc.

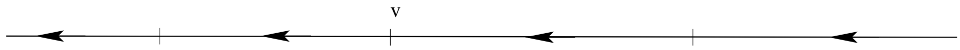

Figure 4: Graph G.

From Proposition 7.1, we deduce that φ solves (E), moreover φ has independent and stationary increments and satisfy the flow property.

In [10], it is proved that flows solutions of (E) defined on graphs like in Figure 3 have modifications which satisfy strong flow properties similar to Proposition 7.1 (2) (see [10, Corollary 2]). Actually on graphs with arbitrary orientation and transmission parameters and such that each vertex has at most two adjacent edges, we can proceed to a direct construction of "global" flows using strong flow properties of "local" flows.

8 Appendix 2 : Complement to Section 5

8.1 A key lemma on BK flow

In this section, we use the same notations as in the beginning of Section 5. Let Y be the BK flow (defined in the previous section) associated to W and $\beta := 2\alpha^+ - 1$. For each $u < v$, let n be the first integer such that $]u, v[$ contains a dyadic number of order n and $f(u, v)$ be the smallest dyadic number of order n contained in $]u, v[$ (f is a deterministic machinery which associates to each $(u, v), u < v$ a dyadic number in $]u, v[$).
For all $s \le t$ and all $(x, y) \in \mathbb{R}^2$, using the convention $\inf \emptyset = +\infty$, set

$$\begin{aligned} T^s_{x,y} &= \inf\{u \ge s,\ Y_{s,u}(x) = Y_{s,u}(y)\}, \\ \tau_s(x) &= \inf\{u \ge s,\ x + W_{s,u} = 0\}, \\ n_{s,t}(x) &= \inf\{n \ge 1,\ Y_{s,t}(x - 2^{-n}) = Y_{s,t}(x + 2^{-n})\}. \end{aligned}$$

For all $s \le t$ and all $x \in \mathbb{R}$, let $n = n_{s,t}(x)$ and define

$$v_{s,t}(x) = f(s, T^s_{x-2^{-n},x+2^{-n}}) \text{ if } \quad t \ge T^s_{x-2^{-n},x+2^{-n}}$$

and $v_{s,t}(x) = 0$ otherwise. Now let $(n, v) = \big(n_{s,t}(x), v_{s,t}(x)\big)$ and define

$$y_{s,t}(x) = f\big(Y_{s,v}(x - 2^{-n}), Y_{s,v}(x + 2^{-n})\big) \text{ if } \quad t \ge T^s_{x-2^{-n},x+2^{-n}}$$

and $y_{s,t}(x) = 0$ otherwise. Note that $(s, t, x, \omega) \longmapsto (v_{s,t}(x, \omega), y_{s,t}(x, \omega))$ is measurable and that for all $s < t$, $(v_{s,t}, y_{s,t})$ is $\mathcal{F}^W_{s,t}$-measurable.

Lemma 8.1. *Let s and x in $\mathbb{R}$. Then a.s. for all $t > \tau_s(x)$, we have*

(i) $n = n_{s,t}(x) < \infty$,
(ii) $v_{s,t}(x) = f(s, T^s_{x-2^{-n},x+2^{-n}})$ *and* $y_{s,t}(x) = f(Y_{s,v}(x - 2^{-n}), Y_{s,v}(x + 2^{-n}))$,
(iii) $Y_{s,t}(x) = Y_{v,t}(y)$, *with* $(v, y) = (v_{s,t}(x), y_{s,t}(x))$.

Proof. See [8, Lemma 3]. □

8.2 Construction of a flow of mappings

In this section, we will use the same notations as in Section 8.1 and in Section 5 with the assumption $\alpha^+ \neq \frac{1}{2}$ if $n \ge 3$ (n is the number of half lines constituting G). Moreover, we set

$$G^+ = \{0\} \cup \cup_{i \in I_+} E_i, \quad G^- = \{0\} \cup \cup_{i \in I_-} E_i.$$

We will review the construction of the unique flow of mappings solving (E) defined on G. Let W be a real white noise. First we will construct $\varphi_{s,\cdot}(x)$ for all $(s,x) \in \mathbb{Q} \times G_{\mathbb{Q}}$ where $G_{\mathbb{Q}} = \{z \in G, |z| \in \mathbb{Q}_+\}$. Denote this set of points by $(s_i, x_i)_{i\geq 0}$ and write $x_i = e_{j_i}(r_i)$ where $r_i \in \mathbb{R}$ and $j_i \in \{1, \cdots, n\}$. Let γ^+, γ^- be two independent random variables respectively taking their values in I_+ and in I_- and such that for $i \in I_+$ and $j \in I_-$,

$$\mathbb{P}(\gamma^+ = i) = \frac{\alpha^i}{\alpha^+} \text{ and } \mathbb{P}(\gamma^- = j) = \frac{\alpha^j}{\alpha^-}.$$

We will construct $\varphi_{s_0,\cdot}(x_0)$, then $\varphi_{s_1,\cdot}(x_1)$ and so on. Let $\mathbb{D}$ be the set of all dyadic numbers on $\mathbb{R}$ and $\{(\gamma_r^+, \gamma_r^-), r \in \mathbb{D}\}$ be a family of independent copies of (γ^+, γ^-) which is also independent of W. If $x = e_i(r)$, recall the definition $\tau_s^x = \tau_s(r)$ where $\tau_s(r)$ is as in the previous paragraph. For $x_0 = e_{j_0}(r_0)$, define $\varphi_{s_0,\cdot}(x_0)$ by

$$\varphi_{s_0,t}(x_0) = \begin{cases} e_{j_0}(r_0 + W_{s_0,t}), & \text{if } s_0 \leq t \leq \tau_{s_0}^{x_0}; \\ 0, & \text{if } t > \tau_{s_0}^{x_0}, Y_{s_0,t}(r_0) = 0; \\ e_h(Y_{s_0,t}(r_0)), & \text{if } \gamma_r^+ = h,\ t > \tau_{s_0}^{x_0}, Y_{s_0,t}(r_0) > 0; \\ e_h(Y_{s_0,t}(r_0)), & \text{if } \gamma_r^- = h,\ t > \tau_{s_0}^{x_0}, Y_{s_0,t}(r_0) < 0, \end{cases}$$

where $r = f(u,v)$ and u, v are respectively the last zero before t and the first zero after t of $Y_{s_0,\cdot}(r)$ (well defined when $Y_{s_0,t}(r_0) \neq 0$). Now, suppose that $\varphi_{s_0,\cdot}(x_0), \cdots, \varphi_{s_{q-1},\cdot}(x_{q-1})$ are defined and let $\{(\gamma_r^+, \gamma_r^-), r \in \mathbb{D}\}$ be a new family of independent copies of (γ^+, γ^-) (that is independent of all vectors (γ^+, γ^-) used until $q-1$ and independent also of W). Let

$$t_0 = \inf\left\{u \geq s_q : Y_{s_q,u}(r_q) \in \{Y_{s_i,u}(r_i), i \in [0, q-1]\}\right\}.$$

Since $t_0 < \infty$, let $i \in [0, q-1]$ and (s_i, r_i) such that $Y_{s_q,t_0}(r_q) = Y_{s_i,t_0}(r_i)$. Now define $\varphi_{s_q,\cdot}(x_q)$ by

$$\varphi_{s_q,t}(x_q) = \begin{cases} e_{j_q}(r_q + W_{s_q,t}), & \text{if } s_q \leq t \leq \tau_{s_q}^{x_q}; \\ 0, & \text{if } \tau_{s_q}^{x_q} < t < t_0, Y_{s_q,t}(r_q) = 0; \\ e_h(Y_{s_0,t}(r_0)), & \text{if } \gamma_r^+ = h,\ \tau_{s_q}^{x_q} < t < t_0, Y_{s_q,t}(r_q) > 0; \\ e_h(Y_{s_q,t}(r_q)), & \text{if } \gamma_r^- = h,\ \tau_{s_q}^{x_q} < t < t_0, Y_{s_q,t}(r_q) < 0; \\ \varphi_{s_i,t}(x_i), & \text{if } t \geq t_0, \end{cases}$$

where r is defined as in $\varphi_{s_0,\cdot}(x_0)$ (from the skew Brownian motion $Y_{s_q,\cdot}(r_q)$). In this way, we construct $(\varphi_{s_i,\cdot}(x_i))_{i\geq 0}$.

Extension. Now we will define entirely φ. Let $s \leq t$ and $x \in G$ be such that $(s,x) \notin \mathbb{Q} \times G_{\mathbb{Q}}$. If $x = e_i(r)$ and $s \leq t \leq \tau_s^x$, define $\varphi_{s,t}(x) = e_i(r + W_{s,t})$. If $t > \tau_s^x$, let m be the first nonzero integer such that $Y_{s,t}(r - 2^{-m}) = Y_{s,t}(r + 2^{-m})$ (when m does not exist we give an arbitrary definition to $\varphi_{s,t}(x)$). Then consider the dyadic numbers

$$v = f\left(s, T^s_{r-2^{-m}, r+2^{-m}}\right), \quad r' = f\left(Y_{s,v}(r - 2^{-m}), Y_{s,v}(r + 2^{-m})\right) \tag{8.1}$$

and finally set $\varphi_{s,t}(x) = \varphi_{v,t}(z)$, where

$$z = e_1(r') \text{ if } r' \geq 0 \text{ and } z = e_{n^+ + 1}(r') \text{ if } r' < 0. \tag{8.2}$$

Note that $\varphi_{s,t}(x, \omega)$ is measurable with respect to (s, t, x, ω).

By [8, Lemma 3], for a "typical" (s,x) a.s. for all $t > \tau_s^x$, m is finite. Note also that : for all $s \leq t$ and all $x = e_i(r) \in G$ a.s.

$$|\varphi_{s,t}(x)| = |Y_{s,t}(r)| \text{ and } \varphi_{s,t}(x) \in G^{\pm} \Leftrightarrow \pm Y_{s,t}(r) \geq 0. \tag{8.3}$$

This is clear when $(s,x) \in \mathbb{Q} \times G_{\mathbb{Q}}$ and remains true for all s,t and x by Lemma 8.1 (iii). The independence of the increments of φ is clear and the stationarity comes from the fact that for all $s \le t$ and $x = e_i(r) \in G$ (even when $(s,x) \in \mathbb{Q} \times G_{\mathbb{Q}}$), if v and r' are defined by (8.1), then on the event $\{t > \tau_s^x\}$, a.s. $\varphi_{s,t}(x) = \varphi_{v,t}(z)$ with z given by (8.2).

Writing Freidlin-Sheu formula [8, Theorem 3] for the Walsh's Brownian motion $t \mapsto \varphi_{s,s+t}(x)$ and using (8.3), we see that φ solves (E).

The flow φ is the unique SFM solving (E) in our case. When $\alpha^+ = \frac{1}{2}$, the BK flow is the trivial flow $x + W_{s,t}$ which is non coalescing. The above construction cannot be applied if $n \ge 3$, no flow of mappings solving (E) can be constructed in this case.

Remark 8.2. *Recall Theorem 5.1 and the lines after (the SFM case). Then* (U^+, U^-) *can be identified with a couple* (γ^+, γ^-) *with law described as above (define* $\gamma^+ = i$ *if* $U^+(i) = 1$ *and* $\gamma^- = j$ *if* $U^-(j) = 1$*). We have seen that working directly with* (γ^+, γ^-) *instead of* (U^+, U^-) *makes the construction more clear.*

8.3 The other solutions.

Suppose $\alpha^+ \neq \frac{1}{2}$ and let m^+ and m^- be two probability measures as in Theorem 5.1. Then, to (m^+, m^-) is associated a SFK K solution of (E) constructed similarly to φ. Let $U^+ = (U^+(i))_{i \in I_+}$ and $U^- = (U^-(j))_{j \in I_-}$ be two independent random variables with respective values in $[0,1]^{n^+}$ and $[0,1]^{n^-}$ such that

$$U^+ \overset{law}{=} m^+, \;\; U^- \overset{law}{=} m^-.$$

In particular a.s. $\sum_{i \in I_+} U^+(i) = \sum_{i \in I_-} U^-(j) = 1$. Let $\{(U_r^+, U_r^-), r \in \mathbb{D}\}$ be a family of independent copies of (U^+, U^-) which is independent of W. Then define

$$K_{s_0,t}(x_0) = \begin{cases} \delta_{e_{j_0}(r_0 + W_{s_0,t})}, & \text{if} \quad s_0 \le t \le \tau_{s_0}^{x_0}; \\ \delta_0, & \text{if} \quad t > \tau_{s_0}^{x_0},\ Y_{s_0,t}(r_0) = 0; \\ \displaystyle\sum_{i \in I_+} U_r^+(i) \delta_{e_i(Y_{s_0,t}(r_0))}, & \text{if} \quad t > \tau_{s_0}^{x_0},\ Y_{s_0,t}(r_0) > 0; \\ \displaystyle\sum_{j \in I_-} U_r^-(j) \delta_{e_j(Y_{s_0,t}(r_0))}, & \text{if} \quad t > \tau_{s_0}^{x_0},\ Y_{s_0,t}(r_0) < 0, \end{cases}$$

where $U_r^+ = (U_r^+(i))_{i \in I_+}, U_r^- = (U_r^-(j))_{j \in I_-}$ and r is the same as in the definition of $\varphi_{s_0,\cdot}(x_0)$. Now K is constructed following the same steps as in the construction of φ.

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